

AD-A060 663

WISCONSIN UNIV-MADISON MATHEMATICS RESEARCH CENTER

F/G 12/1

A COMBINATORIAL PROBLEM ARISING IN THE STUDY OF REACTION-DIFFUS--ETC(U)

MAY 78 J GREENBERG, C GREENE, S HASTINGS

DAA629-75-C-0024

UNCLASSIFIED

MRC-TSR-1850

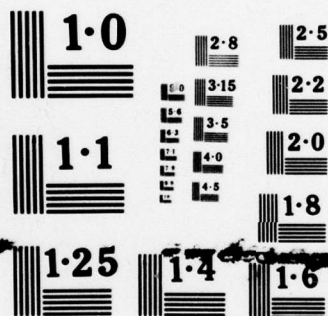
NL

1 OF 1  
ADA  
060663



END  
DATE  
FILMED

1 -79  
DDC



NATIONAL BUREAU OF STANDARDS  
MICROCOPY RESOLUTION TEST CHART

AD A060663

DDC FILE COPY

MRC Technical Summary Report #1850

A COMBINATORIAL PROBLEM ARISING  
IN THE STUDY OF  
REACTION-DIFFUSION EQUATIONS

J. Greenberg, C. Greene,  
and S. Hastings

Mathematics Research Center  
University of Wisconsin-Madison  
610 Walnut Street  
Madison, Wisconsin 53706

May 1978

(Received April 26, 1978)

LEVEL 4

DDC  
RECEIVED  
NOV 1 1978  
RECEIVED

q

A

Approved for public release  
Distribution unlimited

Sponsored by

U. S. Army Research Office  
P. O. Box 12211  
Research Triangle Park  
North Carolina 27709

78 09 12 049

UNIVERSITY OF WISCONSIN - MADISON  
MATHEMATICS RESEARCH CENTER

A COMBINATORIAL PROBLEM ARISING IN THE STUDY OF  
REACTION-DIFFUSION EQUATIONS

J. Greenberg, C. Greene, and S. Hastings

Technical Summary Report #1850

May 1978

ABSTRACT

We study a discrete model based on the observed behavior of excitable media. This model has the basic properties of an excitable medium, that is, a threshold phenomenon, a refractory period, and a globally stable rest point. We are mainly interested in two dimensional periodic patterns. We characterize the initial conditions which lead to such patterns, by introducing a basic invariant, the "winding number of a continuous cycle".

AMS(MOS) Subject Classification: 05B99

Key Words: Pattern formation, Discrete models, Reaction-diffusion.

Work Unit Number 4 - Probability, Statistics, and Combinatorics.

---

Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.

78 09 12 049

UNIVERSITY OF WISCONSIN - MADISON	
MATHEMATICS RESEARCH CENTER	
A COMBINATORIAL PROBLEM ARISING IN THE STUDY OF	
REACTION-DIFFUSION EQUATIONS	
J. Greenberg, C. Greene, and S. Hastings	
Technical Summary Report #1850	
May 1978	
ABSTRACT	
We study a discrete model based on the observed behavior of excitable	
media. This model has the basic properties of an excitable medium, that is,	
a threshold phenomenon, a refractory period, and a globally stable rest point.	
We are mainly interested in two dimensional periodic patterns. We characterize	
the initial conditions which lead to such patterns, by introducing a basic in-	
variant, the "winding number of a continuous cycle".	
AMS(MOS) Subject Classification: 05B99	
Key Words: Pattern formation, Discrete models, Reaction-diffusion.	
Work Unit Number 4 - Probability, Statistics, and Combinatorics.	
Sponsored by the United States Army under Contract No. DAAG29-75-C-0024.	
78 09 12 049	

## SIGNIFICANCE AND EXPLANATION

Pattern formation in living organisms is a basic problem in biology. Recently chemists have been studying certain (inorganic) chemical reactions which lead to interesting temporal and spatial patterns, in an effort to understand how such patterns arise. The interaction of diffusion and chemical reaction effects seems important. In this paper a discrete mathematical model is analyzed to see how these effects can lead to the sorts of behavior which have been seen experimentally.

---

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.



A COMBINATORIAL PROBLEM ARISING IN THE STUDY  
OF REACTION-DIFFUSION EQUATIONS

J. Greenberg, C. Greene, and S. Hastings

The problem we shall consider bears a superficial resemblance to the well-known "game" of Life, as devised by E. J. Conway [1], in which a set of simple rules determines the step-by step evolution of certain patterns in an infinite planar grid. Our problem is also set on an infinite grid of square "cells" and proceeds in discrete time steps. However it differs from its predecessor in having a natural physical interpretation, in terms of reaction-diffusion processes. These are of current interest because of their importance in a variety of biological phenomena, including nerve conduction and morphogenesis. A related paper [2] continues the study of discrete models of such processes which was begun in [3]. However, in our opinion, the combinatorial aspects of the problem have sufficient interest to warrant separate treatment.

To describe our process we label cells  $c = c_{i,j}$  with integer coordinates  $(i,j)$ ,  $-\infty < i,j < \infty$ , and consider an infinite sequence  $t = 0,1,2,3,\dots$  of discrete time steps. To each triple  $(i,j,t)$  associate an integer  $u_{i,j}^t$ , called the "state" of cell  $c_{i,j}$  at time  $t$ . These integers will come from a fixed finite set  $S = \{0,1,2,\dots,N\}$ , where  $N \geq 2$ . The initial states  $u_{i,j}^0$  are chosen arbitrarily from  $S$ . Subsequent states  $u_{i,j}^t$ ,  $t > 0$ , are then determined inductively, according to rules A and B below.

The inductive procedure to be described requires choosing, initially, a fixed integer  $K$  with

$$(1) \qquad 1 \leq K \leq N/2 .$$

The states  $u = 1, 2, \dots, K$  are called "excited", while states  $K + 1, \dots, N$  are called "refractory". Also,  $u = 0$  is sometimes referred to as the "rest" state. The rules for our game are

- A) If  $1 \leq u_{i,j}^t \leq N - 1$ , then  $u_{i,j}^{t+1} = u_{i,j}^t + 1$ , while if  $u_{i,j}^t = N$  then  $u_{i,j}^{t+1} = 0$ .
- B) Suppose that  $u_{i,j}^t = 0$ . To determine  $u_{i,j}^{t+1}$ , examine the four "adjacent" cells  $c_{i',j'}$ , where  $|i - i'| + |j - j'| = 1$ . If one or more of these cells is excited at time  $t$ , then  $u_{i,j}^{t+1} = 1$ . Otherwise,  $u_{i,j}^{t+1} = 0$ .

(FIG. 1)

The motivation for these rules is, roughly, that excitation "diffuses" from an excited region into an adjacent resting region, but not into a refractory region. Also, once a cell is excited, its state evolves according to fixed dynamics with no diffusion effects from neighboring cells, until it returns to rest. We remark that a number of previous authors, starting with Wiener and Rosenblueth [4], have studied similar processes, usually on a computer and again in a biological context. In [2] it is shown how these rules are related to a certain singular limit of some widely studied continuous models of reaction-diffusion processes.

The problem, broadly, is to describe how a given initial pattern  $P_0 = \{(i, j, u_{i,j}^0), -\infty < i, j < \infty\}$  evolves as  $t$  increases. In particular, what sorts of patterns can develop, and will the process continue indefinitely without all cells returning eventually and permanently to rest. In [3] it is observed that for  $N = 2$  and  $K = 1$  there is a complete solution, provided only that the number of non-zero cells at  $t = 0$  is finite. There are two possibilities.

- I. The pattern dies out. By this we mean that for any  $(i, j)$  there is a  $T_{i,j}$  such that  $u_{i,j}^t = 0$  if  $t \geq T_{i,j}$ . Equivalently,

$$\lim_{t \rightarrow \infty} \inf\{|i| + |j| \mid u_{i,j}^t \neq 0\} = \infty.$$

In other words, the pattern becomes identically zero in any finite region in finite time.

II. The pattern persists. Thus there is at least one cell  $c_{i,j}$  such that

$$\{t \mid u_{i,j}^t \neq 0\}$$

is unbounded. Equivalently this is the case for each  $c_{i,j}$ . (This is not hard to show.) Furthermore, the pattern is eventually periodic in any finite region, and can be described as a set of rotating spirals and concentric waves radiating periodically from fixed centers.

(FIG. 2)

In addition, one can easily determine which of I or II will occur, and locate the centers of all rotating spirals and concentric rings, by examining the initial configuration. Since this paper is devoted to  $N > 2$ , and no particular insight is gained by studying  $N = 2$ , we refer the reader to [3] for a more thorough description of the three state model.

In considering the many state version we concentrate on determining whether a pattern will persist or die out. We shall only consider patterns with a finite number of non-zero states at  $t = 0$ . Ideally one would like to find a necessary and sufficient condition for persistence which can be checked at  $t = 0$ . We have not found such a condition, though we do have non-trivial necessary conditions and sufficient conditions of this type. In addition, we give a necessary and sufficient condition for persistence which can be checked after a certain number  $T = T(P_0)$  of iterations have been carried out, where  $T$  depends, roughly, on the size of the initial non-zero set. (See Theorem 5.)

In order to state our results we need a measure of the distance between states in  $S$ . We use the metric  $d( , )$  defined by



$$(2) \quad d(m,n) = \min\{|m - n|, N + 1 - |m - n|\}$$

for any  $m, n$  in  $S$ . Equivalently, identify each  $k$  in  $S$  with the point

$$\hat{k} = e^{\frac{k}{N+1} \cdot 2\pi i}$$

on the unit circle  $C$  in the complex plane. Then

$$d(m,n) = \frac{N+1}{2\pi} \{\text{shorter distance from } \hat{m} \text{ to } \hat{n} \text{ on } C\}.$$

Observe that for any cell  $c_{i,j}$  and any  $t \geq 0$ ,

$$(3) \quad d(u_{i,j}^t, u_{i',j'}^{t+1}) \leq 1.$$

Assume that  $N \geq 3$  and let

$$(4) \quad L = \min\{K + 1, \frac{N+1}{4}\}.$$

Theorem 1. If there is a  $t_0 \geq 0$  such that  $d(u_{u,j}^{t_0}, u_{i',j'}^{t_0}) < L$  whenever  $c_{i,j}$  and  $c_{i',j'}$  are adjacent, then the pattern dies out. (Recall that  $c_{i,j}$  and  $c_{i',j'}$  are adjacent if  $|i - i'| + |j - j'| = 1$ .)

In particular, if the process is persistent, then there must be adjacent cells  $c_{i,j}$  and  $c_{i',j'}$  such that  $d(u_{i,j}^0, u_{i',j'}^0) \geq L$ . In fact, this can be strengthened a bit.

Theorem 2. If the process persists, then there is a fixed pair of adjacent cells  $c_{i,j}$  and  $c_{i',j'}$  such that

$$d(u_{i,j}^t, u_{i',j'}^t) \geq L$$

for all  $t \geq 0$ .

If  $L < \frac{N+1}{4}$ , then eventually an even stronger discontinuity must develop.

Theorem 3. If the process persists, then for any sufficiently large  $t$  there is a pair of adjacent cells  $c_{i,j}$  and  $c_{i',j'}$  such that  $d(u_{i,j}^t, u_{i',j'}^t) > \frac{N+1}{4}$ .

An estimate will be given for when (at what  $t$ ) this inequality must hold.

In order to give sufficient conditions for persistence we introduce the concept of a cycle.

Def. A cycle is an ordered  $(M + 1)$ -tuple  $C = (c^1, c^2, c^3, \dots, c^M, c^{M+1})$  of cells such that  $c^1, \dots, c^M$  are distinct,  $c^{M+1} = c^1$ , and  $c^i$  is adjacent to  $c^{i+1}$  for  $i \leq i \leq M$ .

(FIG. 3)

Def. A cycle  $C$  is said to be continuous at time  $t$  if

$$d(u_i^t, u_{i+1}^t) \leq K \text{ for } 1 \leq i \leq M,$$

where  $u_i^t$  is the state of cell  $c^i$  at time  $t$ .

For such a cycle we then define a "winding number" at time  $t$ . For this purpose recall the previous identification of the states  $k = 0, 1, 2, \dots, N$

with the points  $\hat{k} = e^{\frac{k}{N+1} \cdot 2\pi i}$  on the unit circle. If  $m, n \in S$ , let  $\overrightarrow{mn}$  denote the shorter directed arc from  $\hat{m}$  to  $\hat{n}$ . If both arcs from  $\hat{m}$  to  $\hat{n}$  are of the same length, let  $\overrightarrow{mn}$  be the arc connecting  $\hat{m}$  to  $\hat{n}$  in the counter-clockwise direction. Then, for an ordered pair  $(m, n)$  of integers in  $S$ , let

$$\sigma(m, n) = \begin{cases} d(m, n) & \text{if } \overrightarrow{mn} \text{ connects } \hat{m} \text{ to } \hat{n} \text{ in the counter-clockwise direction} \\ -d(m, n) & \text{otherwise.} \end{cases}$$

The winding number of a continuous cycle  $C$  at time  $t$  is then defined by

$$W_t(C) = \frac{1}{N+1} \sum_{i=1}^M \sigma(u_i^t, u_{i+1}^t).$$

It is not hard to show that  $W_t(C)$  is an integer, and represents the net number of times the unit circle is traversed in the counter-clockwise direction by the points  $\hat{u}_i^t$  as  $i$  runs from 1 to  $M + 1$ . We now give a necessary and sufficient condition for persistence.

Theorem 4. The pattern is persistent if and only if there is a  $T \geq 0$  and a continuous cycle  $C$  at time  $T$  such that  $W_T(C) \neq 0$ .

Obviously this includes a sufficient condition for persistence which can be checked at  $t = 0$ .

It is desirable to find an upper bound for the smallest  $T$  satisfying the conditions in Theorem 4. It can be shown by example that  $T$  may be arbitrarily large if the size of the initial non-zero set is not restricted. Our result in this direction is probably not the best possible.

Theorem 5. Let  $R_1$  be a "diamond" shaped set of the form

$$R_1 = \{c_{i,j} \mid |i| + |j| \leq m + 1\}$$

for some  $m$ , and suppose that

$$u_{i,j}^0 = 0 \text{ if } |i| + |j| > m.$$

Then the pattern persists if and only if there is a continuous cycle with non-zero winding number at time  $T = n(R_1) \cdot K$ , where  $n(R_1)$  is the number of cells in  $R_1$ .

Our final result is proved in almost the same way as Theorem 5.

Theorem 6. The process is eventually periodic in any bounded region.

In other words, there is a  $p$  such that for each  $m$  there is a  $T_m$  with

$$u_{i,j}^{t+p} = u_{i,j}^t$$

whenever  $t \geq T_m$  and  $|i| + |j| \leq m$ .

One can extend these results in several directions. For instance, arrays of cells in more than two dimensions, or with non-rectangular geometry could be considered. Other definitions of "adjacent", or "neighboring" cells might be used. For example in the plane with square cells we could say that  $c_{i,j}$  and  $c_{i',j'}$  are neighbors if  $1 \leq |i - i'| + |j - j'| \leq 2$ , in any of these cases



Theorem 4 goes over without change. However Theorem 1 may need modification. The alternative definition of adjacent cells given above requires the number  $\frac{N+1}{3}$  instead of  $\frac{N+1}{4}$  in Theorem 1. On the other hand, with rectangular geometry and the definition of two cells as adjacent if they have common faces, Theorems 1-6 are essentially unchanged in higher dimensions.

Proofs: A basic observation is that discontinuities in a cycle do not appear spontaneously. This is implied by the following result.

Lemma 1. Suppose that  $c$  and  $d$  are adjacent cells with states  $u^t$  and  $v^t$  at time  $t$ . If

$$(5) \quad d(u^{t_0}, v^{t_0}) \leq K$$

for some  $t_0 \geq 0$ , then

$$d(u^t, v^t) \leq \max\{d(u^{t_0}, v^{t_0}), 1\}$$

for all  $t \geq t_0$ .

Corollary. If a cycle is continuous at  $t_0$ , then it is continuous for all  $t \geq t_0$ .

Proof of Lemma 1. If  $u^{t_0} \neq 0$  and  $v^{t_0} \neq 0$ , then rule B, together with (2), implies that

$$(6) \quad d(u^{t_0+1}, v^{t_0+1}) = d(u^{t_0}, v^{t_0}).$$

We can therefore assume that at least one cell, say  $c$ , is in the resting state at  $t = t_0$ , i.e. that  $u^{t_0} = 0$ . If, in addition,  $v^{t_0} = 0$  then rule B implies that  $d(u^{t_0+1}, v^{t_0+1}) \leq 1$ .

Next suppose that  $u^{t_0} = 0$ ,  $1 \leq v^{t_0} \leq K$ . Again use rule B, to conclude that  $u^{t_0+1} = 1$  and again (6) holds. Finally, if  $K+1 \leq v^{t_0} \leq N$  then (1), (2), and (5) imply that  $v^{t_0} \geq N - K + 1 > N/2$ . But from this it follows that



$$d(v^{t_0+1}, u^{t_0}) = d(v^{t_0}, u^{t_0}) - 1.$$

From (3) and the triangle inequality we get  $d(u^{t_0+1}, v^{t_0+1}) \leq d(u^{t_0}, v^{t_0})$ , completing the proof of Lemma 1.

It turns out that Theorem 4 is the key result so we prove it first.

Proof of Theorem 4. We begin by showing that if  $C$  is a continuous cycle at  $t_0$ , and hence for  $t \geq t_0$ , then  $W_t(C) = W_{t_0}(C)$  for  $t \geq t_0$ . It suffices to show that  $W_{t_0+1}(C) = W_{t_0}(C)$ .

For each pair of integers  $j$  and  $k$  with  $1 \leq j < k \leq M$ , let

$$Q_t(j, k) = \sum_{i=j}^{k-1} \sigma(u_i^t, u_{i+1}^t).$$

In particular,  $W_t(C) = \frac{1}{N+1} Q_t(1, M+1)$ .

Lemma 2. Suppose that  $j$  and  $k$  are integers with  $1 \leq j < k \leq M+1$  and that

$$(7) \quad u_j^t \neq 0, u_k^t \neq 0, \text{ and } u_i^t = 0 \text{ if } j < i < k.$$

Then  $Q_t(j, k) = Q_{t+1}(j, k)$ .

Proof. If  $k = j+1$ , then

$$Q_t(j, k) = \sigma(u_j^t, u_k^t) = \sigma(u_j^{t+1}, u_k^{t+1}) = Q_{t+1}(j, k)$$

because both  $\hat{u}_j^s$  and  $\hat{u}_k^s$  move one step counter-clockwise around the unit circle as  $s$  goes from  $t$  to  $t+1$ .

Now suppose that  $k \geq j+2$ . From (7) we see that

$$u_i^{t+1} = 0 \text{ or } 1 \text{ if } j < i < k.$$

Hence  $\sigma(u_i^{t+1}, u_{i+1}^{t+1}) = u_{i+1}^{t+1} - u_i^{t+1}$  if  $j < i < k-1$  and so

$$Q_{t+1}(j, k) = \sigma(u_j^{t+1}, u_{j+1}^{t+1}) + \sum_{i=j+1}^{k-2} (u_{i+1}^{t+1} - u_i^{t+1}) + \sigma(u_{k-1}^{t+1}, u_k^{t+1}),$$

where the summation term on the right is not present if  $k = j + 2$ . If  $k > j + 2$  then the summation term collapses. We conclude that for any  $k \geq j + 2$ ,

$$(8) \quad Q_{t+1}(j,k) = \sigma(u_j^{t+1}, u_{j+1}^{t+1}) + u_{k-1}^{t+1} - u_{j+1}^{t+1} + \sigma(u_{k-1}^{t+1}, u_k^{t+1}) .$$

From (7) it follows that  $u_j^{t+1}$  and  $u_k^{t+1}$  lie in the set

$$[N - K + 2, N] \cup [0, K + 1] .$$

Since  $u_{k-1}^{t+1}$  and  $u_{j+1}^{t+1}$  are 0 or 1, it is easily seen that

$$\sigma(u_j^{t+1}, u_{j+1}^{t+1}) - u_{j+1}^{t+1} = \sigma(u_j^{t+1}, 0)$$

$$\sigma(u_{k-1}^{t+1}, u_k^{t+1}) + u_{k-1}^{t+1} = \sigma(0, u_k^{t+1})$$

and from (8),

$$\begin{aligned} Q_{t+1}(j,k) &= \sigma(u_j^{t+1}, 0) + \sigma(0, u_k^{t+1}) \\ &= \sigma(u_j^t, 0) + \sigma(0, u_k^t) \\ &= Q_t(j,k) , \end{aligned}$$

where we again use (7). This proves Lemma 2.

The first part of Theorem 4 then follows quickly by letting

$i_1 < i_2 < \dots < i_p$  be those  $i$  such that  $u_i^t \neq 0$ , and observing that

$$W_t(C) = \sum_{\ell=1}^p Q_t(i_\ell, i_{\ell+1}) ,$$

where we set  $i_{p+1} = i_1$ . The desired result follows by applying Lemma 2 to  $j = i_m$ ,  $k = i_{m+1}$  for  $1 \leq m \leq p$ .

Remark. An alternative proof that the winding number of a continuous cycle is constant proceeds as in the following sketch: The states of the cells of  $C$  at time  $t$  can be used to define a continuous map  $F_t : C \rightarrow C$  of the unit circle into itself. To do this, identify the cells  $c^1, \dots, c^M$  of  $C$

with the points  $\hat{c}^j = e^{\frac{j}{M+1} \cdot 2\pi i}$  on  $C$ . Map  $\hat{c}^j$  into  $\hat{u}_j^t$ . Extend this map to a continuous one from  $C$  to  $C$  by interpolation, using shortest arcs along  $C$  and taking the counter-clockwise arc in case of ties. This defines  $F_t$ , and similarly one can define  $F_{t+1}$ , using the states at time  $t+1$ . These two maps are easily seen to be homotopic, and hence have the same winding number.

We next consider the second part of Theorem 4, namely, that if a pattern persists, then eventually there must be a continuous cycle with non-zero winding number. To prove this some additional concepts are helpful.

Definition. A path is an  $M$ -tuple  $P = c^1, \dots, c^M$  of cells such that  $c^i$  is adjacent to  $c^{i+1}$  for  $1 \leq i \leq M-1$ .

Def. A path  $P$  is said to be continuous at time  $t$  if  $d(u_i^t, u_{i+1}^t) \leq K$  for  $1 \leq i \leq M-1$ , where  $u_i^t$  is the state of  $c^i$  at time  $t$ .

By Lemma 1, if  $P$  is continuous at  $t_0$ , then it is continuous for all  $t \geq t_0$ .

We shall say that a cell  $c$  is external to the pattern at time  $t$  if there is a rectangle  $R$  in the plane such that  $c$  is outside  $R$  but all cells which have non-zero states at time  $t$  lie inside  $R$ .

Under the assumption that there are no continuous cycles with non-zero winding number for any  $t \geq 0$ , we can define the potential of a cell  $c_{i,j}$  at time  $t$  for any cell which is connected to an external cell by some continuous path at time  $t$ . Let the path be  $c^1, \dots, c^M$ , where  $c^M = c_{i,j}$  and  $c^1$  is external. Then we set

$$h_t(i,j) = \sum_{k=1}^{M-1} \sigma(u_i^t, u_{i+1}^t).$$

This will be the same for any continuous path connecting  $c_{i,j}$  to any external cell, for otherwise we could find a continuous cycle with non-zero winding



number. All cells external to the initial pattern have a potential for all  $t$ . Also, if  $h_{t_0}(i,j)$  is defined, then so is  $h_t(i,j)$  for  $t \geq t_0$ , and  $h_t(t,j) \geq h_{t_0}(i,j)$ ,  $t \geq t_0$ . Among those cells which are not external to the initial pattern (a finite number), we can allow the possibility that some may never have a potential defined. In any case there must be a  $t_0 \geq 0$  such that no cells have a potential defined for the first time at some  $t \geq t_0$ .

Let  $c_{i,j}$  be any cell with a potential defined for  $t \geq t_0$ . Suppose that for some  $t \geq t_0$ ,  $u_{i,j}^t = 0$  and  $1 \leq u_{i',j'}^t \leq K$  for an adjacent cell  $c_{i',j'}$ . Then  $c_{i',j'}$  has a defined potential at time  $t$  which must be higher than that of  $c_{i,j}$ , since we can connect  $c_{i',j'}$  to the outside by a path going through  $c_{i,j}$ .

Let  $\Delta'$  be the highest potential of any cell at time  $t_0$ . Let  $\Delta$  be the next higher multiple of  $N + 1$ . We claim that no cell can achieve a potential greater than  $\Delta$ . If not, let  $c_{k,l}$  be the first cell to achieve a potential of  $\Delta + 1$ , and suppose this occurs at  $t = t_1$ . Then  $u_{k,l}^{t_1-1} = 0$ , and  $c_{k,l}$  has a neighbor  $c_{k',l'}$  which is excited at time  $t_1 - 1$ , so that  $1 \leq u_{k',l'}^{t_1-1} \leq K$ . But then  $c_{k',l'}$  must have a potential greater than  $\Delta$  at  $t = t_1 - 1$ , which is a contradiction.

But this proves that among all those cells with potential  $\Delta'$  at time  $t_0$ , none can ever become excited again, contradicting the persistence of the pattern.

Proofs of Theorems 1, 2, and 3. Theorems 1, 2, and 3 all follow from Theorem 4.

Notice that as a consequence of Lemma 1, Theorems 1 and 2 are corollaries of Theorem 3. To prove Theorem 3, assume that  $i = K + 1 < \frac{N + 1}{4}$  and choose  $t_0$  large enough to insure that at  $t_0$  there is a continuous cycle  $C = (c^1, \dots, c^M)$  with  $w_{t_0}(C) \neq 0$ . This is possible by Theorem 4. We shall show that



$$d(u_{i,j}^{t_0}, u_{i',j'}^{t_0}) \geq \frac{N+1}{4}$$

for some pair of adjacent cells  $c_{i,j}$  and  $c_{i',j'}$ .

For this purpose we use a different kind of continuity for a cycle. We say that a cycle  $E = (e^1, \dots, e^Q)$  is mildly continuous at  $t_0$  if

$$d(u_i, u_{i+1}) \leq \frac{N+1}{4}$$

for  $1 \leq i \leq Q$ , where  $u_i$  is the state of  $e^i$  at  $t_0$ . (We shall only be concerned with states at  $t = t_0$ , and so we suppress the time in this notation.)

For any mildly continuous cycle  $E$ , the winding number  $w_{t_0}(E)$  is defined just as before. We are assuming that  $w_{t_0}(E) \neq 0$ .

Let  $r = \sup\{d(u_{i,j}^{t_0}, u_{i',j'}^{t_0})/c_{i,j} \mid c_{i,j} \text{ is adjacent to } c_{i',j'}\}$  and suppose that  $r < \frac{N+1}{4}$ .

**Lemma 4.** There is a mildly continuous cycle  $E$  at  $t_0$  which consists of exactly four cells and has non-zero winding number.

**Proof.** Consider  $C$  as an ordered  $M$ -tuple of closed squares in the plane. Let  $\gamma$  be the Jordan curve obtained by joining the centers of consecutive squares of  $C$ . Suppose that the inside of  $\gamma$  contains squares which are not in  $C$ . Such cells we call interior to  $C$ . Then there must be three consecutive cells  $c^i, c^{i+1}, c^{i+2}$  in  $C$  which form three quarters of a square of four cells such that the fourth cell  $c^*$  in this square lies inside  $\gamma$ .

(FIG. 4)

Since  $r < \frac{N+1}{4}$ , the cycle obtained by replacing  $c^{i+1}$  in  $C$  by  $c^*$  is still mildly continuous. Furthermore,

$$\begin{aligned} \sigma(u_i, u_{i+2}) &= \sigma(u_i, u_{i+1}) + \sigma(u_{i+1}, u_{i+2}) \\ &= \sigma(u_i, u^*) + \sigma(u^*, u_{i+2}) \end{aligned}$$

where  $u^*$  is the state of  $c^*$  at  $t_0$ . Therefore the new cycle  $C^*$  has the same (non-zero) winding number as  $C$ . However,  $c^*$  has one less interior cell than  $C$ .

Continuing this shrinking process, we see that there must be a mildly continuous cycle  $C^1$  at  $t_0$  with non-zero winding number and no interior cells. Again there must be consecutive cells  $c^i, c^{i+1}, c^{i+2}$  of  $C^1$  which comprise three quarters of a square of cells, and now these can be chosen so that the fourth cell  $\hat{c}$  in this square is also a cell of  $C^1$ . We can renumber the cycle so that  $\hat{c} = c^j$  for some  $j > i + 2$ .

If  $j = i + 3$ , then there are two cases. The cycle  $\hat{C} = c^i, c^{i+1}, c^{i+2}, c^{i+3}$  may have non-zero winding number, in which case we are done. If, on the other hand,  $\hat{C}$  has winding number 0, then omit  $c^{i+1}$  and  $c^{i+2}$  in  $C^1$  and there results a cycle  $C^2$  with  $w_{t_0}(C^2) = w_{t_0}(C^1)$ . Thus  $C^1$  has been reduced to a still smaller cycle.

Next suppose that  $j > i + 3$ . Proceed according to whether the cycle  $D = (c^{i+2}, c^{i+3}, \dots, c^j)$  has non-zero winding number or not. If  $w_{t_0}(D) \neq 0$ , replace  $C^1$  with  $C$ . If  $w_{t_0}(D) = 0$ , eliminate  $c^{i+2}, \dots, c^j$  from  $C^1$ .

It is thus clear that we arrive eventually at a cycle with the desired properties at  $t_0$ , proving Lemma 4.

But from this result we obtain a contradiction. Since

$$d(u_i, u_{i+1}) \leq r < \frac{N+1}{4},$$

and assuming as we may that  $u_1 = 0$ , we have the inequalities

$$1 \leq u_2 \leq r, u_3 \leq 2r, u_4 \leq 3r.$$

On the other hand,  $u_4 \geq N - r + 1$ . This gives

$$3r \geq N - r + 1, \text{ or } r \geq \frac{N+1}{4},$$

which contradiction proves Theorems 1, 2, and 3.

Proof of Theorems 5 and 6. Let  $R_0$  be the diamond shaped region

$$R_0 = \{c_{i,j} \mid |i| + |j| \leq m\},$$

where  $m$  is chosen so large that  $R_0$  contains the entire non-zero set at  $t = 0$ . Also, let

$$R_n = \{c_{i,j} \mid |i| + |j| \leq m + n\}$$

for  $n = 1, 2, 3, \dots$ .

Lemma 5. Within any given  $R_n$ ,  $n \geq 1$ , the process proceeds independently from the outside of  $R_n$ . More precisely, if  $c_{i,j} \in R^n$  and  $u_{i,j}^{t_0} = 1$ , then  $2 \leq u_{i',j'}^{t_0} \leq K + 1$  for some cell  $c_{i',j'}$  adjacent to  $c_{i,j}$  and also contained in  $R_n$ .

This result implies that if  $\hat{u}_{i,j}^t$  is a second process, obeying the rules A and B if  $c_{i,j} \in R^n$  but following the rule  $u_{i,j}^t = 0$  if  $c_{i,j} \notin R^n$ , then  $\hat{u}_{i,j}^t = \hat{u}_{i,j}^t$  in  $R^n$ . The process  $\hat{u}_{i,j}^t$  is clearly eventually periodic, which leads to our theorem.

Proof of Lemma 5. Suppose, then, that the Lemma is false, and let  $t_0$  be the first time where a cell in some  $R_n$ ,  $n \geq 1$ , is excited by a cell outside  $R_n$ , and not simultaneously excited by a cell in  $R_n$ . Thus there is a cell  $c_{i,j} = c^0$  in  $R_n$  with  $u_{i,j}^{t_0} = u_0^{t_0} = 1$ , while  $2 \leq u_1^{t_0} \leq K + 1$  for some adjacent cell  $c'$  in  $R^{n+1}$ . Furthermore,

$$(8) \quad u_2^{t_0} \notin [2, K + 1]$$

for any cell  $c^2$  in  $R^n$  which is adjacent to  $c^0$ .

However  $u_0^0 - u_1^0 = 0$ , so Lemma 1 implies that  $u_1^{t_0} = 2$ . Also,  $c^1$  must in turn have been excited by a cell  $c^3 \in R^n$ . This follows from our hypotheses on  $t_0$ , since cells of  $R^{n+1}$  which are not in  $R^n$  cannot be adjacent to each other. Then  $u_3^{t_0} = 3$ . Also,  $c^0, c^1$  and  $c^3$  form three cells out of a



square of four cells. Let  $c^2$  be the fourth cell in this square. Then  $c^2 \in R^n$  and  $c^2$  is adjacent to  $c^0$  and  $c^3$ . Since  $u_3^{t_0} = 3$ ,  $u_0^{t_0} = 1$ , and  $d(u_0^{t_0}, u_2^{t_0}) \leq 1$ ,  $d(u_3^{t_0}, u_2^{t_0}) \leq 1$ , it is seen that  $u_2^{t_0} = 2$ , a contradiction of (8). This proves the Lemma.

Theorem 6 is immediate since the process  $\hat{u}_{i,j}^t$ , which is on a finite grid, is clearly periodic. Theorem 5 also follows easily, since  $\hat{u}_{i,j}^t$  can only be persistent if there is a cycle  $c^1, \dots, c^{N+1}$  of cells in  $R_{M+1}$  such that  $1 \leq u_0^1 \leq K$ , and each  $c^i$  excites cell  $c^{i+1}$  within a time  $K$  from when  $c^i$  was excited by  $c^{i-1}$ , for  $2 \leq i \leq N$ . The existence of such a cycle follows easily for a finite grid.

#### REFERENCES

1. Gardner, M., Mathematical Games, Scientific American, Oct. '70, pg. 120; Feb. '71, pg. 112.
2. Greenberg, J. M., B. D. Hassard, and S. P. Hastings, Pattern formation and periodic structures in systems modeled by reaction-diffusion equations, to appear in Bulletin AMS.
3. Greenberg, J. M., and S. P. Hastings, Spatial patterns for discrete models of diffusion in excitable media, to appear in SIAM J. Appld. Math.
4. Wiener, N. and A. Rosenblueth, The mathematical formulation of the problem of conduction of impulses in a network of connected excitable elements, specifically in cardiac muscle, Arch. Inst. Cardiol. Mexico 16, 205-265.



REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER #1850	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER	
4. TITLE (and Subtitle) A COMBINATORIAL PROBLEM ARISING IN THE STUDY OF REACTION-DIFFUSION EQUATIONS,		5. TYPE OF REPORT & PERIOD COVERED Summary Report, no specific reporting period	
7. AUTHOR(s) J. Greenberg, C. Greene S. Hastings		6. PERFORMING ORG. REPORT NUMBER	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Madison, Wisconsin 53706		8. CONTRACT OR GRANT NUMBER(s) DAAG29-75-C-0024	
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office P. O. Box 12211 Research Triangle Park, North Carolina 27709		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Work Unit Number 4 - Probability, Statistics, and Combinatorics	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) 1219p		12. REPORT DATE May 1978	
		13. NUMBER OF PAGES 15	
		15. SECURITY CLASS. (of this report) UNCLASSIFIED	
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report) Approved for public release; distribution unlimited. 14 MRC-TSR-1850			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) Pattern formation, Discrete models, Reaction-diffusion. is studied			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) We study a discrete model based on the observed behavior of excitable media. This model has the basic properties of an excitable medium, that is, a threshold phenomenon, a refractory period, and a globally stable rest point. We are mainly interested in two dimensional periodic patterns. We character- ize the initial conditions which lead to such patterns, by introducing a basic invariant, the winding number of a continuous cycle. are characterized			

221 200